Generalization of the Menger's Theorem to Simplicial Complexes and Certain Invariants of the Underlying Topological Spaces.

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Abstract

We extend the edge version of the classical Menger's Theorem for undirected graphs to n-dimensional simplicial complexes with chains over the field $\mathbb{F}_2$. The classical Menger's Theorem states that two different vertices in and undirected graph can be connected by k different and pairwise edge-disjoint paths if and only if after any deletion of any $k-1$ edges from that graph there still exists a path, which connects these two vertices. We prove that a non-trivial $n-1$ dimensional cycle in an n-dimensional simplicial complex is a boundary of k different and pairwise simplex-disjoint n-dimensional chains over $\mathbb{F}_2$ if and only if after any deletion of any $k-1$ n-dimensional simplices from that complex there still exists an n-dimensional chain whose boundary is that $(n-1)$-dimensional cycle. Using this result we define for $n-1$ dimensional cycles the notion of k-boundance, which is the extension of the classical notion of k-edge-connectivity for pairs of vertices in an undirected graph.

Next we restate both the original Menger's Theorem and our generalization purely in terms of the underlying topological spaces. Thus, k-edge-connectivity and, in general, k-boundance is a property of a pair of points or, in general, of the underlying space of an $(n-1)$- dimensional cycle in the underlying space of the graph, or, more generally, of the n-dimensional simplicial complex. Using these results we produce certain topological invariants of the underlying spaces.

1 Preliminaries - Graphs, Simplicial Complexes and their Underlying Spaces

In this work all the algebra is done over the field $\mathbb{F}_2$. Thus, all our graphs are undirected and all our simplices are non-oriented.

All our graphs and simplicial complexes are finite. We permit multiple edges in our graphs and multiple d-dimensional simplices to correspond to the same $(d + 1)$-tuple of vertices in our simplicial complexes. Our graphs and simplicial complexes are not assumed to be connected. We do not permit loops in our graphs nor repetitions in $(d+1)$-tuples of vertices which correspond to d-dimensional simplices in our complexes. We refer to [3], [2] and [5] for detailed treatment of simplicial complexes and of topology of their underlying spaces.

Thus, we have sets $S_0 = V, S_1 = E, S_2, ..., S_n$ of all simplices of dimension $0, 1, 2, ..., n$ respectively. For each d-dimensional simplex $s \in S_d$, where $d > 0$, we have a unique $(d + 1)$-tuple of different vertices $v_1, ..., v_{d+1}$ from V. The $\mathbb{F}_2$-linear combinations of d-dimensional simplices, including the null linear combination, are called d-dimensional chains. The vector space of all the d-dimensional chains over $\mathbb{F}_2$ is denoted by Ch_d. If $n = 1$ then a simplicial complex is also called an undirected graph.

The boundary map δ is a $\mathbb{F}_2$-linear map from Ch_{d+1} into Ch_d which takes each simplex $s \in S_{d+1}$ to a sum $f_1 + ... + f_{d+2}$ of $d+2$ simplices from S_d so that if to s corresponds the $(d + 2)$-tuple $v_1, ..., v_{d+2}$ of vertices then to each f_i corresponds the $(d + 1)$-tuple obtained by deleting vertex v_i from the original tuple. The boundary map δ from Ch_0 onto $\mathbb{F}_2$ is defined by taking each vertex to 1.

The kernel of δ in Ch_d is denoted by $Cycle_d$ and its elements are called d-dimensional cycles. The image of δ in Ch_d is denoted by $Bound_d$ and its elements are called d-dimensional boundaries. It is a well-known fact that $\delta \circ \delta$ is the zero map. Thus, $Bound_d$ is a subspace of $Cycle_d$ and the quotient space is denoted by H_d and called the d^{th} homology space of the simplicial complex. It is a well known fact that homology spaces are invariants of the underlying topological space of the complex.

Let K be an n-dimensional simplicial complex. It is clear that there are no nontrivial n-dimensional boundaries and so $Cycle_n = H_n$. Every $(n - 1)$-dimensional simplex f of K appears as a summand in the boundary of m different n-dimensional simplices $s_1, ..., s_m$ of

K. Here m could be zero. We say that the degree $deg(f)$ of f in K is m and we say that f is a face of the simplices $s_1, ..., s_m$.

The points x of the underlying topological space $X(K)$ of the complex K, which correspond to a point of some d-dimensional simplex s of K, but do not correspond to any points of any faces of s are, by abuse of notations, called the inner points of s.

- Every inner point x of an n-dimensional simplex has a neighborhood U in $X(K)$ such that exists a homeomorphism between U and the space $\mathbb{R}^n$, which takes x to the origin.
- Every inner point x of an $(n-1)$-dimensional simplex f with $deg(f) = 0$ has a neighborhood U in $X(K)$ such that exists a homeomorphism between U and the space $\mathbb{R}^{n-1}$, which takes x to the origin.
- Every inner point x of an $(n-1)$-dimensional simplex f with $deg(f) > 0$ has a neighborhood U in $X(K)$ such that exists a homeomorphism between U and $deg(f)$ copies of the closed half-space $\mathbb{R}^{n+}$, all glued together along their boundary, which takes x to their common origin.

Clearly, exists a homeomorphism between 2 copies of the closed half-space $\mathbb{R}^{n+}$, glued together along their boundary, and the space $\mathbb{R}^n$, which takes the origin to the origin.

For a point $x \in X(K)$ we say that the degree $deg(x)$ of x is m if exists a neighborhood U of x in $X(K)$ and a homeomorphism between U and m copies of the closed half-space $\mathbb{R}^{n+}$, all glued together along their boundary, which takes x to their common origin. If for a point x exists a neighborhood U of x in $X(K)$ and a homeomorphism between U and the space $\mathbb{R}^{n-1}$, which takes x to the origin, we say that $deg(x) = 0$. Note, that for some points in $X(K)$ the notion of degree can be undefined.

Thus every point of $X(K)$, which corresponds to an inner point of an $(n-1)$-dimensional simplex of degree d, must itself be of degree d. Every point of $X(K)$, which corresponds to an inner point of an n-dimensional simplex, must be of degree 2. Vice versa, if a point of $X(K)$ has degree d then if $d \neq 2$ then this point corresponds to an inner point of an $(n-1)$-dimensional simplex of degree

d and if $d = 2$ then it corresponds either to an inner point of an $(n-1)$-dimensional simplex of degree 2 or to an inner point of an n-dimensional simplex.

Definition 1.1. For $d = 0, 1, ...$ the closure in $X(K)$ of the set of all the points of $X(K)$ of degree d is denoted by $X_d(K)$.

Definition 1.2. For $d = 2, 3, ...$ the closure in $X(K)$ of the set of all the points of $X(K)$ of degree greater than or equal to d is denoted by $Y_d(K)$. Clearly $Y_d(K) = \bigcup_{i=d}^{\infty} X_i(K)$

Definition 1.3. For $d = 2, 3, ...$ the closure in $X(K)$ of the set of all the points of $X(K)$ of degree greater than or equal to d is denoted by $Y_d(K)$. Clearly $Y_d(K) = \bigcup_{i=d}^{\infty} X_i(K)$

Definition 1.4. For an n-dimensional simplicial complex K we call the $(n-1)$-dimensional simplicial complex K', which is obtained from K by deleting all n-dimensional simplices and all $(n-1)$-dimensional simplices of degree 2, the irregularity skeleton of K.

Clearly, $Y_3(K)$ is a subspace of $X(K')$. Here $X(K')$ is the underlying topological space of K'.

2 Preliminaries - Undirected Graphs, k-edge-connectivity, the edge version of the Menger's Theorem

We briefly review these important notions and results in their classical form - for the detailed treatment of this topic we refer to [1] and [6]: Let G be an undirected graph and u, v be two vertices of G. A path from u to v is an alternating sequence $P = (w_0, e_1, w_1, ..., w_{t-1}, e_t, w_t)$ of vertices and edges such that $w_0 = u$ and $w_t = v$ and that to each edge e_i the corresponding pair of vertices is w_{i-1}, w_i and that if $i \neq j$ then $e_i \neq e_j$. We say that the length of the path P is the number t of edges in it. If $u = v$ then a such path is called a cycle. We permit the trivial cycle $C = u$ of length 0 from u to itself. Since in this work we do not care about the direction of the path and since every path from u to v can be taken in the reversed order and will become a path from v to u, we

abuse the notations as speak of P as of a path between u and v.

We say that two different vertices u and v are k-edge-connected in G if exist k paths $P_1, ..., P_k$ between u and v such that no two of these paths have any common edges. By definition, every u is k-edge-connected to itself for all k. If u and v are 1-edge-connected in G they are also called connected in G.

The edge version of the Menger's Theorem states that u and v are k-edge-connected if and only if after any deletion of any $k-1$ or less edges from the graph G the vertices u and v will always remain connected in it. In other words u and v are k-edge-connected if and only if any deletion of any $k-1$ edges does not break the connectivity of u and v.

Now we define the notion of a homological path and restate the definition of the k-edge-connectivity and the Menger's Theorem in a different language, which permits us in the following Section to generalize these notions from graphs to simplicial complexes:

A homological path between vertices u and v in an undirected graph G is a 1-dimensional chain P such that its boundary $\delta(P)$ is $u+v$. A homological cycle, or a homological cycle between any vertex and itself, is just any element of $Cycle_1$ of the graph. Note, that any path P between two vertices the sum of its edges is a homological path between the same vertices. In the opposite direction we have that every homological path between two vertices contains as some of its summands all the edges from some actual path between these two vertices.

We say that two vertices u and v are k-edge-connected if exist k homological paths $P_1, ..., P_k$ between u and v such that any two of these homological paths have no common summands in them. It is easy to check that both our definitions follow from each-other. In the case when $u = v$ all k paths are equal to the trivial empty 1-dimensional chain, since the trivial 1-dimensional chain has no common summands with itself.

Definition 2.1. A list is a set with possible repetitions of elements. A sublist L' of a list L is a collection of some of the elements of the

list L, taken with any repetitions.

Thus, for example, we have a list $L = a, a, b, c, c, c$ and a sublist $L' = a, a, a, b, b, b, b, b$. In this work, unless stated otherwise, we always require that our sublists have the same amount of elements, counting with repetitions, as the original lists. Thus a sublist L' of $L = a, a, b, c, c, c$, unless stated otherwise, will be required to have 6 elements in it, counting with repetitions.

Abusing the notation, we refer to pairs $a - b$ of vertices a and b of an undirected graph G as 0-dimensional cycles of G. Indeed, any such pair produces a 0-dimensional cycle $a + b$. Here if $a = b$ then for any a the produced cycle is the trivial cycle. Let $L = a_1 - b_1, a_2 - b_2, ..., a_k - b_k$ be a list of k pairs of vertices. For any vertex a of G we permit the pair $a - a$ in a list L. As already mentioned above, any repetitions of pairs is permitted in a list L.

Definition 2.2. A list $L = a_1 - b_1, a_2 - b_2, ..., a_k - b_k$ be of k pairs of vertices of G is called k-boundant if exist k homological paths $P_1, ..., P_k$ in G with no common summands in any two of them, such that $\delta(P_1), ..., \delta(P_k)$ is a sublist L' of the list L.

It is easy to check that two different vertices u and v are k-edge-connected in G if and only if the list $L = u - v, u - v, ..., u - v$ is k-boundant. The following theorem is a somewhat generalized restatement of the edge version of the classical Menger's Theorem. Indeed, Menger's Theorem easily follows from it:

Theorem 2.3. *A list L of pairs of vertices of G is k-boundant in G if and only if after any deletion of any $k - 1$ or less edges from G the list L will remain at least 1-boundant. In other words, after any deletion of any $k - 1$ or less edges from G for at least one pair $a_i - b_i$ from L the 0-dimensional cycle $a_i + b_i$ will be a boundary of some 1-dimensional chain of G.*

In the next Section we will state and prove a much more general theorem about k-boundance of $(n - 1)$-dimensional cycles in n-dimensional simplicial complexes, from which the current theorem will directly follow.

3 Simplicial Complexes - k-boundance and the generalization of the Menger's Theorem

Let K be an n-dimensional simplicial complex and $L = c_1, c_2, ..., c_k$ be a list of k $(n-1)$-dimensional cycles of K. The trivial empty cycle as well as any repetitions of cycles are permitted in L. A sublist L' of L is a list of k cycles, counting with repetitions, taken L.

Definition 3.1. The list L is called k-boundant in K if exist k n-dimensional chains $P_1, ..., P_k$ with no two of them having any common summands, such that $\delta(P_1), ..., \delta(P_k)$ is a sublist L' of L.

By definition, an empty n-dimensional chain does not have any common summands with itself. Hence if one of the cycles in a list L is trivial then L is k-boundant for all k.

Now we state and prove the generalization of the edge version of the Menger's Theorem to simplicial complexes:

Theorem 3.2. *The list L is k-boundant if and only if after any deletion of any $k-1$ or less n-dimensional simplices from K the list L will remain at least 1-boundant. In other words, after any deletion of any $k-1$ or less n-dimensional simplices from K at least one cycle c_i from L will be a boundary of some n-dimensional chain of K.*

Proof. One direction is obvious - if $P_1, ..., P_k$ are n-dimensional chains such that any n-dimensional simplex of K appears as a summand at most in one of them then any deletion of any $k-1$ or less n-dimensional simplices from K will not ruin all of these chains and so at least one cycle from L will stay a boundary of some chain. We prove the other direction.

Assume that we are wrong. Then exist some counter-examples. Let K be one of the counter-examples with the least amount of n-dimensional simplices in it. Due to minimality, every n-dimensional simplex w of K must belong to some (at least one) set of k different n-dimensional simplices, deletion of all of which from K will prevent all the cycles from the list L from being boundaries in K (otherwise n-dimensional simplex w can be dropped from K and we would still get a counter-example - contradiction to the minimality of K).

Let $w_1, ..., w_k$ be a set of k different n-dimensional simplices of K such that in the complex K_1, obtained by deleting $w_1, ..., w_k$ from K, all the cycles from L are not boundaries. Then for each n-dimensional simplex w_i, for $i = 1, ..., k$, exists some (at least one) n-dimensional chain P_i of K such that $\delta(P_i) = c_i$ for some cycle c_i from L and such that the deletion of all the other n-dimensional simplices $w_1, ..., w_{i-1}, w_{i+1}, ..., w_k$ from K does not effect the chain P_i. In other words, w_i is a summand in P_i and all the other w_j, where $j \neq i$, are not summands in P_i.

For all $i = 1, ..., k$ we define a new $(n-1)$-dimensional cycle $u_i = c_i + \delta(w_i)$ of K. Clearly all $u_1, ..., u_k$ belong to the simplicial complex K_1. Now we define a new list $L_1 = u_1, ..., u_k$ of k $(n-1)$-dimensional cycles of K_1 and we define n-dimensional chains $Z_i = P_i + w_i$ for $i = 1, ..., n$. Since none of Z_i contains any of w_j, $Z_1, ..., Z_k$ are chains in the complex K_1.

In K_1 any deletion of any $k-1$ or less n-dimensional simplices will preserve at least one of the cycles from $\bar{L}$ as a boundary of some n-dimensional chain of K_1. Otherwise that same deletion of these same n-dimensional simplices in K will prevent from all the cycles from the original list L from being boundaries in K.

Hence, by our minimality assumption of K, the list L_1 is k-boundant in K_1 and exist k n-dimensional chains $J_1, ..., J_k$ in K_1, with no two of them having any common summands, such that $\delta(J_1), ..., \delta(J_k)$ is a sublist of L_1. We define n-dimensional chains $\Upsilon_i = J_i + w_i$ of K for $i = 1, ..., k$.

It is easy to see that for $i \neq j$ the chains Y_i and Y_j have no common summands. Moreover, it is easy to see that $\delta(\Upsilon_1), ..., \delta(\Upsilon_k)$ is a sublist of L. This implies that the list L is k-boundant and this is a contradiction to the fact that K is a counter-example. $\square$

We recommend to compare this proof with the proof of the edge version of the original Menger's Theorem in [4].

Definition 3.3. An $(n-1)$-dimensional cycle c of K is called k-boundant in K in the list $L = c, ..., c$ of k copies of c is k-boundant

in K.

Definition 3.4. Two $(n-1)$-dimensional cycles c_1 and c_2 of K are called k-cobordant in K if the list L composed of k repartitions of the cycle $c_1 + c_2$ is k-cobordant in K.

It follows from this definition that any $(n-1)$-dimensional cycle is k-cobordant to itself for any k. An important consequence of this generalized Menger's theorem is:

Corollary 3.5. *Being k-cobordance is an equivalence relation on the $(n-1)$-dimensional cycles in an n-dimensional simplicial complex K.*

Proof. Note that if the sum c_1+c_2 of two $(n-1)$-dimensional cycles c_1 and c_2 is a boundary of an n-dimensional chain P_1 and the sum $c_2 + c_3$ of two $(n-1)$-dimensional cycles c_2 and c_3 is a boundary of an n-dimensional chain P_2 then the sum $c_1 + c_3$ is, clearly, the boundary of the an n-dimensional chain $P_1 + P_2$.

The only non-obvious part, which we prove now, is that if c_1 is k-cobordant to c_2 in K and c_2 is k-cobordant to c_3 in K then c_1 is k-cobordant to c_3 in K. But any deletion of any $k-1$ or less n-dimensional simplices from K will not effect at least one n-dimensional chain P' whose boundary is $c_1 + c_2$. Nor will that deletion effect at least one n-dimensional chain P'' whose boundary is $c_2 + c_3$. Hence any deletion of any $k-1$ or less n-dimensional simplices from K will not effect at least one n-dimensional chain $P'+P''$ whose boundary is c_1+c_3. So, by the generalized Menger's theorem, c_1 and c_3 are k-cobordant in K. ☐

4 k-boundance and certain related Topological invariants of the underlying space

Let K be an n-dimensional simplicial complex. It is easy to see that if an $(n-1)$-dimensional cycle c of K is a boundary in K then every $(n-1)$-dimensional simplex f, which appears as a summand in c, must be a summand in the boundary of at least one n-dimensional simplex of K. Thus $deg(f)$ is at least 1. Moreover, if c is k-boundant then every $(n-1)$-dimensional simplex f, which appears as a summand in c, must be a summand in the boundary of

at least k different n-dimensional simplex of K. Thus $deg(f)$ is at least k. Thus, any k-boundant $(n-1)$-dimensional cycle c of K, for any $k > 2$, must belong to $Cycle_{n-1}(K')$, where K' is the irregularity skeleton of K. Moreover, the underlying topological space $X(c)$ of c must be a subspace of $Y_k(K)$.

Note that since K' is an $(n-1)$-dimensional simplicial complex, there is a natural isomorphism between $Cycle_{n-1}(K')$ and $H_{n-1}(K')$. Thus we can speak of the underlying spaces $X(h)$ of the elements h of $H_{n-1}(K')$, which are all defined as subspaces of $X(K')$.

Definition 4.1. Let c be a $(n-1)$-dimensional cycle of K. We call the underlying space $X(c)$ of c a topological cycle of $X(K)$.

By definition, we will always say that any topological cycle is 0-boundant in $X(K)$. Let P be an n-dimensional chain of K.

Definition 4.2. Let P be any n-dimensional chain of K. We define the set of simplices of various dimensions $\bar{P}$ as the set of all the n-dimensional simplices, which appear as summands in P, and of all the faces of these simplices, which do not appear as faces of any other n-dimensional simplices of K, and the set of all the faces of these faces, which do not appear as faces of any other $(n-1)$-dimensional simplices of K and so on.

Definition 4.3. Let P be any n-dimensional chain of K and c be any $(n-1)$-dimensional cycles of K. Simplicial complex $K - P + c$ is defined as K with all the simplices of $\bar{P}$, which are not summands in c nor faces of these summands nor faces of these faces and so on, deleted from it.

Definition 4.4. A topological cycle $X(c)$ is called k-boundant in $X(K)$ if exists an n-dimensional chain P of K such that $\delta(P) = c$ and such that in the topological space $X(K - P + c)$ the cycle $X(c)$ is $(k-1)$-boundant.

Notice that $X(K-P+c)$ is precisely the closure of $X(K)-X(P)$ in $X(K)$, union with $X(c)$.

Lemma 4.5. *An $(n-1)$-dimensional cycle c of K' is k-boundant in K if and only if its corresponding topological cycle $X(c)$ is k-boundant in $X(K)$.*

Proof. The proof in both directions is straightforward. $\square$

From this Lemma and our generalization of the Menger's Theorem it follows that a topological cycle $X(c)$ is k-boundant in $X(K)$ if and only if after any deletion of $k-1$ or less n-dimensional simplices from K topological cycle $X(c)$ will remain 1-boundant in $X(K)$. Of course, for a topological cycle $X(c)$ being 1-boundant in $X(K)$ just means that c represents the zero element of $H_{n-1}(K)$.

The topological invariants are obtained as follows:

Definition 4.6. The subspace Γ of $H_{n-1}(Y_3(K))$ is the kernel of the map from $H_{n-1}(Y_3(K))$ into $H_{n-1}(X(K))$ which is induced by the imbedding of $Y_3(K)$ into $X(K)$.

Definition 4.7. For each $k = 3, 4, ...$ we define Γ_k to be the space of all the elements h in Γ such that their underlying topological space $X(h)$ in $Y_3(K)$ is k-cobordant in $X(K)$.

From the above arguments it follows that:

Lemma 4.8. *The $\mathbb{F}_2$-vector spaces Γ_k, for all $k = 3, 4, ...$, are invariants of the space $X(K)$ and do not depend on a particular simplicial complex K.*

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